Name of College - S.S. College Jehanabad [illegible]

Part - Mathematics

Topic - Problems based on Pedal Equation of a Polar Curve

Class - (B.Sc [illegible])

Time - 10.00 am to 11.00 am

11.05 am to 11.45 am

Date - 22 - 07 - 2020

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1. Method for finding the pedal equation of a curve from its polar equation -

Step I - Write down the polar equation of the curve $f(r, \theta) = 0$ ——— (i)

Find

p = length of the perpendicular from the pole on the tangent at (r, θ)

Step II $\quad p = r\sin\phi$ ——— (ii)

Step III $\quad \tan\phi = r\dfrac{d\theta}{dr}$ ——— (iii)

Step IV - Eliminate θ and ϕ between the equations (i) (ii) and (iii)

we shall get the required pedal equation

Problems :- Find the equation of the curve

$$r^m = a^m \cos m\theta$$

By the question

Solution :- Equation of Curve

$$r^m = a^m \cos m\theta$$

$$\Rightarrow m \log r = m \log a + \log \cos m\theta$$ (Taking log on both sides)

Diff. with respect to θ

$$\frac{m}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos m\theta}(-\sin m\theta)\cdot m$$

$$\frac{m}{r}\frac{dr}{d\theta} = -\frac{m \sin m\theta}{\cos m\theta}$$

$$\frac{1}{r}\frac{dr}{d\theta} = -\tan m\theta$$

$$\Rightarrow r\frac{d\theta}{dr} = -\cot m\theta$$

$$\Rightarrow \tan \phi = \tan(\pi/2 + m\theta)$$

$$\therefore \phi = \pi/2 + m\theta$$

Now $p = r \sin \phi$

$$= r \sin\left(\frac{\pi}{2} + m\theta\right)$$

Now $p = r \sin \phi = r \sin\left(\frac{\pi}{2} + m\theta\right)$

$$= r \cos m\theta = r \frac{r^m}{a^m}$$ $\left(\because r^m = a^m \cos m\theta \Rightarrow \frac{r^m}{a^m} = \cos m\theta \right)$

Hence the required pedal Equation is

$$p = \frac{r^{m+1}}{a^m}$$

Example 2. Find the pedal equation of the curve

$$2a = r(1+\cos\theta)$$

Solution:→ By the question
Equation of the curve

$$2a = r(1+\cos\theta)$$

$$\Rightarrow \frac{1}{r}\cdot 2a = 1+\cos\theta$$

Diff. w.r.t. respect to θ, we get—

$$-\frac{1}{r^2}\cdot 2a\frac{dr}{d\theta} = -\sin\theta$$

$$\Rightarrow \frac{1}{r^2}\cdot 2a\frac{dr}{d\theta} = \sin\theta$$

$$\Rightarrow \frac{dr}{d\theta} = \frac{r^2\sin\theta}{2a}$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{2a}{r^2\sin\theta}$$

But $\tan\phi = r\frac{d\theta}{dr}$

$$= r\cdot\frac{2a}{r^2\sin\theta}$$

$$= \frac{2a}{r\sin\theta}$$

Since $2a = r(1+\cos\theta)$

$$\Rightarrow \frac{2a}{r} = 1+\cos\theta$$

$$\therefore \tan\phi = \frac{1+\cos\theta}{\sin\theta} = \frac{\cancel{2}\cos^2\theta/2}{\cancel{2}\sin\theta/2\cos\theta/2} = \frac{\cos\frac{\theta}{2}}{\sin\theta/2}$$

$$= \cot\theta/2 = \tan(\pi/2-\theta/2)$$

Therefore

$$\phi = \frac{\pi}{2} - \frac{\theta}{2}$$

Also $p = r\sin\phi$

$$= r\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$= r\cos\theta/2$$

or $p^2 = r^2\cos^2\theta/2$

$$= r^2\,\frac{1+\cos\theta}{2}$$

$\left[\text{Since } 2a = r(1+\cos\theta) \Rightarrow \frac{2a}{r} = 1+\cos\theta\right]$

$\therefore p^2 = \frac{r^2}{2} \times \frac{2a}{r} = ar$

$\therefore$ required Pedal Equation

$$p^2 = ar$$

Problem 3: For the Cardioide $r = a(1-\cos\theta)$, Prove the following results :—

(i) $\phi = \frac{\theta}{2}$ (ii) $p = 2a\sin^3\theta$

(iii) $p^2 = \frac{r^3}{2a}$

Solution :→ Here By the question equation of curve is

$r = a(1-\cos\theta)$ —(1) $\Rightarrow \frac{r}{a} = 1-\cos\theta$

Diff. with respect to θ

$\frac{dr}{d\theta} = a \times \sin\theta \Rightarrow \frac{d\theta}{dr} = \frac{1}{a\sin\theta}$

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Now $\tan\phi = r\dfrac{d\theta}{dr}$

$= r \times \dfrac{1}{a\sin\theta}$

Putting $r = a(1-\cos\theta)$

$\Rightarrow \tan\phi = \dfrac{a(1-\cos\theta)}{a\sin\theta} = \dfrac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2} = \tan\frac{\theta}{2}$

$\therefore \phi = \frac{\theta}{2}$

(ii) Since $p = r\sin\phi$

$= r \times \sin\frac{\theta}{2}$

$= a(1-\cos\theta) \times \sin\frac{\theta}{2}$

$= a \times 2\sin^2\frac{\theta}{2} \times \sin\frac{\theta}{2}$

$= 2a\sin^3\theta/2$

(iii) Since $p = r\sin\phi$

$\Rightarrow p^2 = r^2\sin^2\phi$

$= r^2\sin^2\theta/2$

$= r^2 \cdot \sin^2\theta/2$

Since $r = a(1-\cos\theta)$

$\Rightarrow \dfrac{r}{a} = 1-\cos\theta = 2\sin^2\theta/2$

$\Rightarrow \dfrac{r}{2a} = \sin^2\theta/2$

$\therefore p^2 = \dfrac{r^2 \times r}{2a} = \dfrac{r^3}{2a}$

Problem :- In a Parabola Page 6

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$\frac{2a}{r} = 1-\cos\theta$, Prove the following

results :- (i) $\varphi = \pi - \frac{\theta}{2}$ (ii) $p = \frac{a}{\sin\theta/2}$

(iii) $p^2 = ar$ (iv) Polar Subtangent $= 2a\operatorname{cosec}\theta$.

Solution $\rightarrow$

Here By the question, Equation of Parabola is

$$\frac{2a}{r} = 1-\cos\theta$$

Diff. with respect to θ

$$-\frac{2a}{r^2}\frac{dr}{d\theta} = \sin\theta = 2\sin\theta/_2\cos\theta/2$$

$$\Rightarrow -\frac{\cancel{2}a}{r^2}\frac{dr}{d\theta} = \cancel{2}\sin\theta/_2\cos\theta/2$$

$$\Rightarrow -\frac{a}{r^2}\frac{dr}{d\theta} = \sin\theta/_2\cos\theta/_2 \Rightarrow \frac{dr}{d\theta} = -\frac{r^2\sin\theta/_2\cos\theta/_2}{a}$$

Since we know $p = r\sin\varphi$ $\Rightarrow \frac{d\theta}{dr} = -\frac{a}{r^2\sin\theta/_2\cos\theta/_2}$

and $r\frac{d\theta}{dr} = \tan\varphi$

$$\therefore \tan\varphi = r\frac{d\theta}{dr} = \frac{2a}{1-\cos\theta}$$

[Since $\frac{2a}{r} = 1-\cos\theta = 2\sin^2\theta/_2$

$\Rightarrow r = \frac{2a}{\cancel{4}\sin^2\theta/_2} = \frac{a}{\sin^2\theta/_2}$

$\Rightarrow r^2 = \frac{a^2}{\sin^4\theta/_2}$

$$= \frac{2a}{2\sin^2\theta/_2} \times \frac{(-a)}{r^2\sin\theta/_2\cos\theta/2}$$

$$= -\frac{\cancel{2}\cancel{a^2}}{\cancel{2}\sin^{\cancel{2}}\theta/2} \cdot \frac{1}{\frac{\cancel{a^2}}{\cancel{\sin^4\theta/2}} \times \cancel{\sin\theta}/_2\cos\theta/2}$$

$$= -\frac{\sin\theta/_2}{\cos\theta/_2} = -\tan\theta/_2 = \tan(\pi-\theta/_2)$$

$$\therefore \varphi = \pi - \theta/_2$$

Again $p = r\sin\varphi = r\sin(\pi-\theta/_2) = r\sin\theta/_2$ $\therefore$

$$= \frac{a}{\sin^2\theta/_2} \times \sin\theta/2 = \frac{a}{\sin\theta/_2}$$

$$\cancel{p^2 = \frac{a^2}{\sin^2\theta/_2} = \frac{a^2}{\sin(\pi-\theta/2)}} \quad p^2 = r^2/\sin^2\theta/_2$$

$\therefore \frac{2a}{r} = 1 - \cos\theta = 2\sin^2\theta/2$

$\Rightarrow \frac{a}{r} = \sin^2\theta/2$

$\therefore p^2 = r^2 \sin^2\varphi$

$= r^2 \cdot \left[\sin\left(\pi - \theta/2\right)\right]^2$

$= r^2 \sin^2\theta/2$

$= r^2 \cdot \frac{a}{r} = ar$

Now polar sub tangent

$= r^2 \frac{d\theta}{dr}$

$= \frac{2a}{\sin\theta} = \underline{2a\,\operatorname{cosec}\theta}$